A Multi Period Equilibrium Pricing Model*

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Abstract

In this paper, we propose an equilibrium pricing model in a dynamic multi-period stochastic framework with uncertain income streams. In an incomplete market, there exist two traded risky assets (e.g. stock/commodity and weather derivative) and a non-traded underlying (e.g. temperature). The risk preferences are of exponential (CARA) type with a stochastic coefficient of risk aversion. Both time consistent and time inconsistent trading strategies are considered. We obtain the equilibriums prices of a contingent claim written on the risky asset and non-traded underlying. By running numerical experiments we examine how the equilibriums prices vary in response to changes in model parameters.

Keywords: Time inconsistent control, incomplete market, equilibrium price.

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1 Introduction

Hitherto, there has been an increasing literature on pricing contingent claims written on non-tradable underlyings in a dynamic multi-period equilibrium framework. One example of such contingent claim is a weather derivative, in which case the underlying is the temperature process. One approach in pricing this financial instruments is to use a multi-period stochastic equilibrium model. In financial economics there is a big literature on this issue.

Rubinstein (1976) considers a multi-period state-preference equilibrium model without explicit modelling of production/investment. Brennan (1979) looks at a multiperiod equilibrium problem in which the representative agent exhibits constant risk aversion. Bhattacharya (1981) extends the model of Rubinstein (1976) to show that risk/return tradeoffs are linear relations linking instantaneous expected returns of assets to the instantaneous covariances of returns with aggregate consumption. Bizid and Jouini (2001) derives restrictions on the equilibrium state-price deflators independent on the choices of utility function in an incomplete market. Camara (2003) obtains preference-free option prices in a discrete equilibrium model where representative agent has exponential utility and aggregate wealth together with the underlying asset price have transformed normal distributions.

Our paper presents a partial equilibrium model with two exogenous assets, one tradable and one non-tradable. A derivative security, written on the tradable and non-tradable assets, is priced in equilibrium by a representative agent who receives unspanned random income within an incomplete multi-period market. Cao and Wei (2004), Lee and Oren (2009), Lee and Oren (2010), Cheridito et al (2011) are related to our work. Cao and Wei (2004) generalizes the model of Lucas (1978) to provide an equilibrium framework for valuing weather derivatives in a multi-period setting. Lee and Oren (2009) explores a single-period equilibrium pricing model in a multi commodity setting and mean-variance preferences. Lee and Oren (2010) is a follow up in a multi-period framework. Cheridito et al (2011) establishes results on the existence and uniqueness of equilibrium in dynamically incomplete financial markets with preferences of monetary type and heterogeneous agents.

In our model the representative agent has risk preferences of exponential type, time and state dependent, with time changing coefficient of risk aversion. Inspired by Gordon and St-Amour (2000), we assume that the risk aversion coefficient is a stochastic process. Gordon and St-Amour (2000) motivates this change by the fact that it can explain asset-price movements which fixed preference paradigm can not explain. Lately, the issue of time changing risk aversion received some attention in the financial literature. For instance, Barberis (2001) considers a model in which the loss aversion depends on prior gains and losses, so it may change through time. Danthine et al (2004) allows the representative agent's coefficient of relative risk aversion to vary with the underlying economy's growth rate. Gordon and St-Amour (2004) explains equity premium puzzle by state-dependent risk preferences. Yuan and Chen (2006) shows that dynamic risk aversion plays a critical role in the dynamics of asset price fluctuations.

A time changing risk preference leads to time inconsistent investment strategies. It means that an investor may have an incentive to deviate from the optimal strategies which he/she computed at some past time. In order to overcome this issue, Bjork and Murgoci (2010) develops a theory for stochastic control problems which are time inconsistent in the

sense that they do not admit a Bellman optimality principle; they consider the subgame perfect Nash equilibrium strategies as a substitute for the optimal strategies.

This paper considers both time consistent and time inconsistent optimal strategies. Time consistent optimal strategies are the subgame perfect strategies. Time inconsistent optimal strategies are the classical optimal strategies given that the agent does not update his/her risk preferences. Time consistent (inconsistent) equilibrium price is defined by imposing the market clearing condition for time consistent (inconsistent) optimal strategies.

In the present work the exogenous assets have stochastic drifts and volatilities. It may be that they depend on each other (in a weather model we consider a commodity whose volatility is influenced by the temperature process). A derivative security is priced in equilibrium within this model. Our main result is an iterative algorithm which yields the equilibrium prices (time consistent and time inconsistent). At each stage the equilibrium prices depend on the current risk aversion level, and all previous wealth and risk aversion levels. The algorithm constructs recursively one period pricing kernels. Moreover, the time inconsistent equilibrium pricing kernel equals the marginal utility. We prove that the equilibrium pricing measures are martingale measures so the equilibrium prices (time consistent and time inconsistent) are arbitrage free. Numerical experiments shed light into the structure of equilibriums prices. We show that utility indifference prices and equilibriums ones are different. The utility indifference price was introduced by Hodges and Neuberger (1989). By now, there are several papers on this topic; we recall only a few. [1] Pirvu and Zhang (2011) derives utility indifference prices in a model with time changing risk aversion.

Next, we consider examples in which the non-traded underlying affects the income, thus creating an incentive for the agent to hold the derivative in order to hedge the risk. Our plots show that equilibriums prices are increasing in risk aversion, fact explained by an increased hedging demand. We add a stochastic unspanned component to the income and this slightly decreases the equilibrium prices. This is explained by a decrease in the hedging demand, which is a consequence of income being only partial affected by the non-traded underlying. Finally, in a regime switching model we explore the effect of changing risk aversion. Here we find that an increase of twenty percent in risk aversion can cause a percentage change in the time consistent equilibrium price (with the benchmark being the time inconsistent equilibrium price) anywhere between minus four and seventeen percent.

The optimal strategies are obtained by backward induction. First order conditions together with the market clearing give the equilibrium prices. We consider a partial equilibrium model because of the problem we want to address. However our method can be easily extended to a full equilibrium model (in fact the computations are simpler in that case).

The remainder of this paper is organized as follows. Section 2 presents the model. Section 3 provides the equilibrium pricing valuation. Numerical experiments are presented in the section 4. Proofs of the results are delegated to an appendix.

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[1] on discrete time Musiela and Zariphopoulou (2004), Musiela et al (2009); on continuous time Henderson (2002), Musiela and Zariphopoulou (2004); for an overview see Henderson Hobson (2004).

2 The Model

We consider a multi-period stochastic model of investment. The trading horizon is $[0,T]$, with T a exogenous finite horizon. There are $N+1$ trading dates: $t_n = nh$, for $n = 0,1,\cdots,N$, and $h = \frac{T}{N}$. Let $(b^1, b^2, ..., b^d) := (b^1_{t_n}, b^2_{t_n}, ..., b^d_{t_n})_{n=0,1,...,\infty}$, be a d-dimensional binomial random walk on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_{t_n}\}, \mathbb{P})$. The random walk is assumed symmetric under $\mathbb{P}$ in the sense that

$$\mathbb{P}(\Delta b^i_{t_n} = \pm 1) = 1/2, \qquad i = 1,2,...,d. \tag{2.1}$$

There are three securities available for trading in our model; a riskless bond, a primary asset (e.g. stock or commodity) and a derivative security. We take the bond as numeraire, thus it can be assumed to offer zero interest rate. The primary asset price process $C := \{C_{t_n}; n = 0,1,\ldots,N\}$, follows the difference equation :

$$\begin{cases} \Delta C_{t_n} = C_{t_n}(\mu^c_{t_n} h + \sigma^c_{t_n}\sqrt{h}\Delta b^1_{t_n}), \quad n = 0,1,...N-1 \\ \\ C_0 = c > 0, \end{cases} \tag{2.2}$$

for some adapted drift process $\mu^c := \{\mu^c_t; t = 0, h \ldots, (N-1)h, Nh\}$ and volatility process $\sigma^c := \{\sigma^c_t; t = 0, h \ldots, (N-1)h, Nh\}$ which are chosen so that the commodity price remains positive. The derivative security is written on the primary asset and a non-tradable underlying. The non-traded asset $S := \{S_{t_n}; n = 0,1,\ldots,N\}$, follows the difference equation :

$$\begin{cases} \Delta S_{t_n} = S_{t_n}(\mu^s_{t_n} h + \sigma^s_{t_n}\sqrt{h}(\rho\Delta b^1_{t_n} + \sqrt{1-\rho^2}\Delta b^2_{t_n})), \quad n = 0,1,...,N-1 \\ S_0 = s > 0, \end{cases} \tag{2.3}$$

for some adapted drift process $\mu^s := \{\mu^s_t; t = 0, h \ldots, (N-1)h, Nh\}$, volatility process $\sigma^s := \{\sigma^s_t; t = 0, h \ldots, (N-1)h, Nh\}$, and a correlation coefficient ρ, with $|\rho| < 1$. The two dimensional process $P = (C, S)$, exogenously given, is referred to as the forward process.

The derivative security $D := \{D_{t_n}; n = 0,1,\ldots,N\}$ is to be priced in equilibrium. Therefore we have a partial equilibrium model. It is motivated by a situation in which the primary asset and derivative are priced in different markets. Take for example energy and weather derivatives. Although correlated (in California it was observed a high correlation between energy prices and temperature process) energy and weather derivatives are priced within different markets.

2.1 Trading strategies

Let α_{t_n} be the wealth invested in the primary asset at time t_n, and β_{t_n} the number of shares of derivative held at time t_n; denote $\pi_{t_n} := \{\alpha_{t_n}, \beta_{t_n}\} \in \mathcal{F}_{t_n}, n = 0,1\ldots,N-1$. The value of a self-financed portfolio satisfies the following stochastic difference equation:

$$\Delta X^\pi_{t_n} \;:=\; \alpha_{t_n}(\mu^c_{t_n} h + \sigma^c_{t_n}\sqrt{h}\Delta b^1_{t_n}) + \beta_{t_n}(\Delta D_{t_n} + \varphi(t_n, C_{t_n}, S_{t_n})h). \tag{2.4}$$

Here $\varphi(t_n, C_{t_n}, S_{t_n})h$ is the dividend paid for holding the stock on $[t_n, t_{n+1}]$. At maturity, $t_N := T = Nh$, the representative agent in this economy receives random income I_{t_N}, which is $\mathcal{F}_T$ adapted. Thus, his/her final wealth is

$$W^{\pi}_{t_N} = X^{\pi}_{t_N} + I_{t_N}. \tag{2.5}$$

The random income may depend on all the random walks $\{b^1, b^2, ..., b^d\}$, so it may not be spanned by the existing assets. This makes our market model incomplete.

2.2 Risk Preferences

The representative agent utility is assumed to be of exponential type, time and state dependent. Moreover, the coefficient of absolute risk aversion is a stochastic process γ_{t_n}, $\{\mathcal{F}_{t_n}\}$ adapted, $n = 0, 1 \ldots, N-1$. More precisely

$$U(x, t_n, \omega) = -\exp(-\gamma_{t_n}(\omega)x). \tag{2.6}$$

This modeling approach is not new; in the introduction we pointed out a number of papers which consider time changing risk aversion. The performance of an investment strategy π is measured by the above expected utility criterion applied to the final wealth. At time t_n the optimization criterion is

$$\sup_{\pi \in \Pi_{t_n}} \mathbb{E}[-e^{-\gamma_{t_n} W^{\pi}_{t_N}} | \mathcal{F}_{t_n}]. \tag{2.7}$$

Here $n = 0, 1 \ldots, N-1$, $W^{\pi}_{t_N}$ is given by (2.5), and Π_{t_n} denotes the set of admissible trading strategies,

$$\Pi_{t_n} := \{\pi_{t_n}, \pi_{t_{n+1}}, \cdots, \pi_{t_{N-1}} : \pi_{t_k} \in \mathcal{F}_{t_k}, \text{ such that } \mathbb{E}|X^{\pi}_{t_k}| < \infty, k = n, n+1 \ldots N-1\}. \tag{2.8}$$

2.3 Optimal Time Inconsistent Strategies

They are the classical optimal strategies given that the risk preferences are not updated. More precisely $\hat{\pi} \in \Pi_{t_0}$ is an optimal time inconsistent strategy if it satisfies

$$\hat{\pi} = \arg \sup_{\pi \in \Pi_{t_0}} \mathbb{E}^{\mathbb{P}}[-\exp(-\gamma_{t_0} W^{\pi}_{t_N}) | \mathcal{F}_{t_0}]. \tag{2.9}$$

They are called time inconsistent because they fail to remain optimal at later times t_n, in the sense that

$$\hat{\pi} \neq \arg \sup_{\pi \in \Pi_{t_n}} \mathbb{E}^{\mathbb{P}}[-\exp(-\gamma_{t_n} W^{\pi}_{t_N}) | \mathcal{F}_{t_n}]. \tag{2.10}$$

2.4 Optimal Time Consistent Strategies

In this section, we introduce the optimal time consistent strategies defined as subgame perfect strategies. First, let us consider the time period $[(N-1)h, Nh]$ (recall that $t_N = T = Nh$). At time $(N-1)h$ consider the optimization problem:

$$(P1) \qquad \sup_{\pi \in \Pi_{t_{N-1}}} \mathbb{E}[-\exp(-\gamma_{t_{N-1}} W_{t_N}^{\pi}) | \mathcal{F}_{t_{N-1}}]. \tag{2.11}$$

In our model sup in (P1) is attained and we denote

$$\pi_{t_{N-1}}^* = \arg \max_{\pi \in \Pi_{t_{N-1}}} \mathbb{E}[-\exp(-\gamma_{t_{N-1}} W_{t_N}^{\pi}) | \mathcal{F}_{t_{N-1}}]. \tag{2.12}$$

On the time period $[(N-2)h, Nh]$, restrict to the trading strategies π be of the form:

$$\pi = \begin{cases} \pi_{t_{N-1}}^*, & \text{on } [(N-1)h, Nh), \\ \\ \pi_{t_{N-2}}, & \text{on } [(N-2)h, (N-1)h), \end{cases} \tag{2.13}$$

for an arbitrary $\mathcal{F}_{t_{N-2}}-$ adapted control $\pi_{t_{N-2}}$ such that $(\pi_{t_{N-2}}, \pi_{t_{N-1}}^*) \in \Pi_{t_{N-2}}$; consider the optimization problem

$$(P2) \qquad \sup_{\pi \in \Pi_{t_{N-2}}} \mathbb{E}[-\exp(-\gamma_{t_{N-2}} W_{t_N}^{\pi}) | \mathcal{F}_{t_{N-2}}]. \tag{2.14}$$

In our model sup in (P2) is attained and we denote

$$(\pi_{t_{N-2}}^*, \pi_{t_{N-1}}^*) = \arg \max_{\pi \in \Pi_{t_{N-2}}} \mathbb{E}[-\exp(-\gamma_{t_{N-2}} W_{t_N}^{\pi}) | \mathcal{F}_{t_{N-2}}]. \tag{2.15}$$

Further we proceed iteratively. On the time period $[(N-n)h, Nh]$ one restricts to trading strategies π of the form:

$$\pi = \begin{cases} \pi_{t_k}^*, & \text{for } k = N-(n-1), N-(n-2), \cdots, N-1, \\ \\ \pi_{t_k} & \text{for } \ k = N-n, \end{cases} \tag{2.16}$$

for an arbitrary $\mathcal{F}_{t_{N-n}}-$ adapted control $\pi_{t_{N-n}}$ such that $(\pi_{t_{N-n}}, \pi_{t_k}^*)_{\{k=N-n+1,\cdots,N-1\}} \in \Pi_{t_{N-n}}$. Consider the optimization problem

$$(Pn) \qquad \max_{\pi \in \Pi_{t_{N-n}}} \mathbb{E}[-\exp(-\gamma_{t_{N-n}} W_{t_N}^{\pi}) | \mathcal{F}_{t_{N-n}}]. \tag{2.17}$$

The sup in (Pn) is attained and we denote

$$(\pi_{t_{N-n}}^*, \pi_{t_{N-n+1}}^*, \cdots, \pi_{t_{N-1}}^*) = \arg \max_{\pi \in \Pi_{t_{N-n}}} \mathbb{E}[-\exp(-\gamma_{t_{N-n}} W_{t_N}^{\pi}) | \mathcal{F}_{t_{N-n}}]. \tag{2.18}$$

The optimal time consistent strategy is $\pi^* = (\pi_{t_0}^*, \pi_{t_1}^*, \cdots, \pi_{t_N}^*)$.

2.5 Time Consistent versus Time Inconsistent Strategies

Let us recall that time inconsistencies in this model are due to time changing risk aversion. Indeed in the case of constant risk aversion, the optimal time consistent and time inconsistent strategies coincide, i.e. $\pi^* = \hat{\pi}$. In general, it is hard to show that the optimal time consistent strategy outperforms the optimal time inconsistent strategy. We show that this is the case if the risk preferences are updated in the following two period model. Let us assume that $I_{t_2} = 0$, and only one asset is available for trading (the primary asset with constant drift and volatility). It is claimed that

$$\mathbb{E}[-\exp(-\gamma_{t_1} W_{t_2}^{\pi^*})|\mathcal{F}_{t_1}] \geq \mathbb{E}[-\exp(-\gamma_{t_1} W_{t_2}^{\hat{\pi}})|\mathcal{F}_{t_1}]. \tag{2.19}$$

Indeed in this model it can be shown that $\pi^*_{t_1} = \hat{\pi}_{t_1}$, and hence $W_{t_1}^{\pi^*} = W_{t_1}^{\hat{\pi}}$. By the definition of optimal time consistent strategies, (2.19) yields.

3 Equilibrium Valuation

We assume that there exists a representative agent with risk preferences given by (2.6). The representative agent trades in C and D in order to maximize expected utility of his/her final wealth. This can be achieved by the optimal time inconsistent strategy $\hat{\pi} = (\hat{\alpha}, \hat{\beta})$ if the representative agent does not update his/her risk preferences. Otherwise, the optimal time consistent strategy $\pi^* = (\alpha^*, \beta^*)$ will be used. Thus, depending on weather or not the representative agent updates his/her risk preferences we introduce two notions of equilibriums: time consistent and time inconsistent. They are given by the market clearing condition in the formal definition below.

Definition 3.1 *Given the terminal payoff D_{t_N} and the dividend stream $\varphi(t_n, C_{t_n}, S_{t_n})h$, D_{t_n} is a time consistent equilibrium price if and only if*

$$\beta^*_{t_n} = 1,$$

for every $n = 0, 1, ..., N-1$. Likewise, D_{t_n} is a time inconsistent equilibrium price if and only if

$$\hat{\beta}_{t_n} = 1,$$

for every $n = 0, 1, ..., N-1$.

This simply says that there is one unit of derivative in the market and it is priced such that "it is optimal" (time consistent or time inconsistent) for the representative agent to acquire it. The interest of the representative agent in holding the derivative comes from the fact that his/her income is exposed to the non-traded asset S; thus, the risk of income fluctuations due to S can be hedged by trading D. Let $r^c_{t_n}$, given by

$$r^c_{t_n} := \frac{\mu^c_{t_n}}{\sigma^c_{t_n}}, \tag{3.1}$$

be the market price of risk (MPR) for the primary asset which is assumed positive. We choose time length h small enough such that $1 \geq r^c_{t_n}\sqrt{h}$.

3.1 Single Period

In this case the optimal time consistent and time inconsistent strategies coincide and so do the corresponding equilibriums. Define $A_{t_{N-1}} := \{\Delta b^1_{t_{N-1}} = 1\}$ and $A^c_{t_{N-1}} := \{\Delta b^1_{t_{N-1}} = -1\}$.

Theorem 3.1 *The equilibrium price at time t_{N-1} is given by*

$$D_{t_{N-1}} = \mathbb{E}^{Q^*}[D_{t_N} + \varphi(t_{N-1}, C_{t_{N-1}}, S_{t_{N-1}})h|\mathcal{F}_{t_{N-1}}],$$

where the probability measure Q^* is defined by

$$\frac{dQ^*}{d\mathbb{P}} = \Lambda_{t_N}\mathbb{E}[\frac{dQ^*}{d\mathbb{P}}|\mathcal{F}_{t_{N-1}}].$$

The pricing kernel Λ_{t_N} is

$$\Lambda_{t_N} := \begin{cases} \lambda_{t_{N-1}} \frac{e^{-\gamma_{t_{N-1}}[D_{t_N}+I_{t_N}]}}{\mathbb{E}[e^{-\gamma_{t_{N-1}}[D_{t_N}+I_{t_N}]}|A_{t_{N-1}} \vee \mathcal{F}_{t_{N-1}}]}, & \text{if} \quad \omega \in A_{t_{N-1}} \\ \lambda_{t_{N-1}} \frac{e^{-\gamma_{t_{N-1}}[D_{t_N}+I_{t_N}]}}{\mathbb{E}[e^{-\gamma_{t_{N-1}}[D_{t_N}+I_{t_N}]}|A^c_{t_{N-1}} \vee \mathcal{F}_{t_{N-1}}]}, & \text{if} \quad \omega \in A^c_{t_{N-1}}, \end{cases} \tag{3.2}$$

with

$$\lambda_{t_{N-1}} = \begin{cases} 1 - r^c_{t_{N-1}}\sqrt{h}, & \text{if} \quad \omega \in A_{t_{N-1}} \\ 1 + r^c_{t_{N-1}}\sqrt{h}, & \text{if} \quad \omega \in A^c_{t_{N-1}}. \end{cases} \tag{3.3}$$

The optimal trading strategy is given by

$$\begin{aligned} \alpha^*_{t_{N-1}} &= \frac{1}{2\gamma_{t_{N-1}}\sigma^c_{t_{N-1}}\sqrt{h}} \log\left(\frac{1 + r^c_{t_{N-1}}\sqrt{h}}{1 - r^c_{t_{N-1}}\sqrt{h}}\right) \\ &+ \frac{1}{2\gamma(J_{t_{N-1}})\sigma^c_{t_{N-1}}\sqrt{h}} \log\left(\frac{\mathbb{E}[e^{-\gamma_{t_{N-1}}[D_{t_N}+I_{t_N}]}|A_{t_{N-1}} \vee \mathcal{F}_{t_{N-1}}]}{\mathbb{E}[e^{-\gamma_{t_{N-1}}[D_{t_N}+I_{t_N}]}|A^c_{t_{N-1}} \vee \mathcal{F}_{t_{N-1}}]}\right). \end{aligned}$$

Proof of this Theorem is done in Appendix A.

□

Next we prove that the probability measure Q^* is a martingale measure so the equilibrium prices are arbitrage free.

Lemma 3.2 *If the dividend $\varphi = 0$, then the traded assets $\{C_{t_n}\}_{n=N-1,N}$ and $\{D_{t_n}\}_{n=N-1,N}$ are martingales under Q^*.*

Proof: $\{D_{t_n}\}_{n=N-1,N}$ is martingale under Q^* by definition. Next we show that $\{C_{t_n}\}_{n=N-1,N}$ is martingale under Q^*. It suffices to prove that with $n = N-1$

$$\begin{aligned}
\mathbb{E}^{Q^*}[\frac{C_{t_{n+1}}}{C_{t_n}}|\mathcal{F}_{t_n}] &= \mathbb{E}^{Q^*}[1+\sigma^c_{t_n}\sqrt{h}(r^c_{t_n}\sqrt{h}+\Delta b^1_{t_n})|\mathcal{F}_{t_n}] \\
&= 1+\sigma^c_{t_n}\sqrt{h}\mathbb{E}^{Q^*}[(r^c_{t_n}\sqrt{h}+\Delta b^1_{t_n})|\mathcal{F}_{t_n}] \qquad (3.4)\\
&= 1. \qquad (3.5)
\end{aligned}$$

This is the case if

$$\mathbb{E}^{Q^*}[(r^c_{t_n}\sqrt{h}+\Delta b^1_{t_n})|\mathcal{F}_{t_n}] = 0. \tag{3.6}$$

When $n = N-1$, (3.6) is equivalent to

$$\frac{r^c_{t_{N-1}}\sqrt{h}+1}{2}\mathbb{E}\left[\Lambda_{t_N}\mathbb{E}[\frac{dQ^*}{d\mathbb{P}}|\mathcal{F}_{t_{N-1}}]|A_{t_{N-1}}\vee\mathcal{F}_{t_{N-1}}\right]+\frac{r^c_{t_n}\sqrt{h}-1}{2}\mathbb{E}\left[\Lambda_{t_N}\mathbb{E}[\frac{dQ^*}{d\mathbb{P}}|\mathcal{F}_{t_{N-1}}]|A^c_{t_{N-1}}\vee\mathcal{F}_{t_{N-1}}\right]=$$

$$\frac{1-(r^c_{t_{N-1}})^2h}{2}\mathbb{E}[\frac{dQ^*}{d\mathbb{P}}|\mathcal{F}_{t_{N-1}}]+\frac{(r^c_{t_{N-1}})^2h-1}{2}\mathbb{E}[\frac{dQ^*}{d\mathbb{P}}|\mathcal{F}_{t_{N-1}}]=0,$$

so the claim yields.

□

3.2 Multiple Periods

Let us define the sets $A_{t_{N-n}} := \{\Delta b^1_{t_{N-n}} = 1\}$ and $A^c_{t_{N-n}} := \{\Delta b^1_{t_{N-n}} = -1\}$; to ease notations, let us denote

$$\mathbb{E}_{t_n}[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_{t_n}],\ \mathbb{E}_{t_n}[\cdot|\mathcal{G}] := \mathbb{E}[\cdot|\mathcal{G}\vee\mathcal{F}_{t_n}],$$

for every $\mathcal{G}\subset\mathcal{F}$. The equilibrium prices are computed by a recursive algorithm. Imagine that they were found at all prior times and now we want to find them at t_{N-n}. Define the random variables $Y^*_{t_{N-(n-1)}}$ and $\hat{Y}_{t_{N-(n-1)}}$ by:

$$e^{-\gamma_{t_{N-n}}Y^*_{t_{N-(n-1)}}} := \mathbb{E}_{t_{N-n+1}}\left[e^{-\gamma_{t_{N-n}}[\sum_{k=N-(n-1)}^{N-1}\Delta X^*_{t_k}+I_{t_N}]}\right], \tag{3.7}$$

and

$$e^{-\gamma_{t_{N-n}}\hat{Y}_{t_{N-(n-1)}}} := \mathbb{E}_{t_{N-n+1}}\left[e^{-\gamma_{t_0}[\sum_{k=N-(n-1)}^{N-1}\Delta\hat{X}_{t_k}+I_{t_N}]}\right]. \tag{3.8}$$

Here

$$\Delta X^*_{t_k} = \alpha^*_{t_k}(\mu^c_{t_k} h + \sqrt{h}\sigma^c_{t_k}\Delta b^1_{t_k}) + \Delta D_{t_k} + \varphi(t_k, C_{t_k}, S_{t_k})h,$$

for any $k = N-n+1, \cdots, N-2, N-1$; the optimal time consistent strategy $\alpha^*_{t_k}$ is given by

$$\begin{aligned} \alpha^*_{t_k} &= \frac{1}{2\gamma_{t_k}\sigma^c_{t_k}\sqrt{h}} \log\left(\frac{1+r^c_{t_k}\sqrt{h}}{1-r^c_{t_k}\sqrt{h}}\right) \\ &+ \frac{1}{2\gamma_{t_k}\sigma^c_{t_k}\sqrt{h}} \log\left(\frac{\mathbb{E}_{t_k}[e^{-\gamma_{t_k}[D_{t_{k+1}}+Y^*_{t_{k+1}}]}|A_{t_k}]}{\mathbb{E}_{t_k}[e^{-\gamma_{t_k}[D_{t_{k+1}}+Y^*_{t_{k+1}}]}|A^c_{t_k}]}\right). \end{aligned} \tag{3.9}$$

Moreover

$$\Delta \hat{X}_{t_k} = \hat{\alpha}_{t_k}(\mu^c_{t_k} h + \sqrt{h}\sigma^c_{t_k}\Delta b^1_{t_k}) + \Delta D_{t_k} + \varphi(t_k, C_{t_k}, S_{t_k})h,$$

for any $k = N-n+1, \cdots, N-2, N-1$; the optimal time inconsistent strategy $\hat{\alpha}_{t_k}$ is given by by

$$\begin{aligned} \hat{\alpha}_{t_k} &= \frac{1}{2\gamma_{t_0}\sigma^c_{t_k}\sqrt{h}} \log\left(\frac{1+r^c_{t_k}\sqrt{h}}{1-r^c_{t_k}\sqrt{h}}\right) \\ &+ \frac{1}{2\gamma_{t_0}\sigma^c_{t_k}\sqrt{h}} \log\left(\frac{\mathbb{E}_{t_k}[e^{-\gamma_{t_0}[D_{t_{k+1}}+\hat{Y}_{t_{k+1}}]}|A_{t_k}]}{\mathbb{E}_{t_k}[e^{-\gamma_{t_0}[D_{t_{k+1}}+\hat{Y}_{t_{k+1}}]}|A^c_{t_k}]}\right). \end{aligned} \tag{3.10}$$

Notice that $Y^*_{t_{N-(n-1)}}$ and $\hat{Y}_{t_{N-(n-1)}}$ are the certainty equivalents (time consistent and time inconsistent) at time $n-1$. In the special case of constant coefficient of absolute risk aversion they are equal. Next, define the one step period pricing kernels $\Lambda^*_{t_{N-n+1}}$ and $\hat{\Lambda}_{t_{N-n+1}}$ by

$$\Lambda^*_{t_{N-n+1}} := \begin{cases} \lambda_{t_{N-n}} \dfrac{e^{-\gamma_{t_{N-n}}[D_{t_{N-n+1}}+Y^*_{t_{N-n+1}}]}}{\mathbb{E}_{t_{N-n}}[e^{-\gamma_{t_{N-n}}[D_{t_{N-n+1}}+Y^*_{t_{N-n+1}}]}|A_{t_{N-n}}]}, & \text{if} \quad \omega \in A_{t_{N-n}} \\ \lambda_{t_{N-n}} \dfrac{e^{-\gamma_{t_{N-n}}[D_{t_{N-n+1}}+Y^*_{t_{N-n+1}}]}}{\mathbb{E}_{t_{N-n}}[e^{-\gamma_{t_{N-n}}[D_{t_{N-n+1}}+Y^*_{t_{N-n+1}}]}|A^c_{t_{N-n}}]}, & \text{if} \quad \omega \in A^c_{t_{N-n}}, \end{cases} \tag{3.11}$$

$$\hat{\Lambda}_{t_{N-n+1}} := \begin{cases} \lambda_{t_{N-n}} \dfrac{e^{-\gamma_{t_0}[D_{t_{N-n+1}}+\hat{Y}_{t_{N-n+1}}]}}{\mathbb{E}_{t_{N-n}}[e^{-\gamma_{t_0}[D_{t_{N-n+1}}+\hat{Y}_{t_{N-n+1}}]}|A_{t_{N-n}}]}, & \text{if} \quad \omega \in A_{t_{N-n}} \\ \lambda_{t_{N-n}} \dfrac{e^{-\gamma_{t_0}[D_{t_{N-n+1}}+\hat{Y}_{t_{N-n+1}}]}}{\mathbb{E}_{t_{N-n}}[e^{-\gamma_{t_0}[D_{t_{N-n+1}}+\hat{Y}_{t_{N-n+1}}]}|A^c_{t_{N-n}}]}. & \text{if} \quad \omega \in A^c_{t_{N-n}}, \end{cases} \tag{3.12}$$

Here

$$\lambda_{t_{N-n}} = \begin{cases} 1 - r^c_{t_{N-n}}\sqrt{h}, & \text{if} \quad \omega \in A_{t_{N-n}} \\ 1 + r^c_{t_{N-n}}\sqrt{h}, & \text{if} \quad \omega \in A^c_{t_{N-n}}. \end{cases} \tag{3.13}$$

The following Theorem is the main result of the paper.

Theorem 3.2 *The time consistent equilibrium price at time t_{N-n} is given by*

$$D_{t_{N-n}} = \mathbb{E}^{Q^*}_{t_{N-n}}[D_{t_{N-n+1}} + \varphi(t_{N-n}, C_{t_{N-n}}, S_{t_{N-n}})h],$$

where the probability measure Q^* is defined by

$$\frac{dQ^*}{d\mathbb{P}} = \Lambda^*_{t_N}\Lambda^*_{t_{N-1}}\ldots\Lambda^*_{t_1}.$$

The optimal time consistent strategy (in the primary asset) is $\alpha^* = (\alpha^*_{t_0}, \alpha^*_{t_1}, \cdots, \alpha^*_{t_N})$, with $\alpha^*_{t_k}$ defined by (3.9) . The time inconsistent equilibrium price at time t_{N-n} is given by

$$D_{t_{N-n}} = \mathbb{E}^{\hat{Q}}_{t_{N-n}}[D_{t_{N-n+1}} + \varphi(t_{N-n}, C_{t_{N-n}}, S_{t_{N-n}})h],$$

where the probability measure $\hat{Q}$ is defined by

$$\frac{d\hat{Q}}{d\mathbb{P}} = \hat{\Lambda}_{t_N}\hat{\Lambda}_{t_{N-1}}\ldots\hat{\Lambda}_{t_1}.$$

The optimal time inconsistent strategy (in the primary asset) is $\hat{\alpha} = (\hat{\alpha}_{t_0}, \hat{\alpha}_{t_1}, \cdots, \hat{\alpha}_{t_N})$, with $\hat{\alpha}_{t_k}$ defined by (3.10) .

Proof of this Theorem is done in Appendix B.

□

For the time inconsistent equilibrium price, we recover the following classical result.

Corollary 3.3 *The time inconsitent pricing kernel equals the marginal utility, i.e.,*

$$\hat{\Lambda}_{t_{N-n+1}} = \frac{\mathbb{E}_{t_{N-n+1}}[U'(\hat{W}_{t_N})]}{\mathbb{E}_{t_{N-n}}[U'(\hat{W}_{t_N})]}, \tag{3.14}$$

where $U(x) = -e^{-\gamma x}$, and $\hat{W}_{t_N}$ (see (2.5)) is the optimal time inconsistent wealth.

Proof of this Corollary is done in Appendix C.

□

Lemma 3.4 *If the dividend $\varphi = 0$, then the traded assets $\{C_{t_n}\}_{n=0,1,\ldots,N}$ and $\{D_{t_n}\}_{n=0,1,\ldots N}$ are martingales under Q^*.*

Proof: The proof is similar to Lemma 3.2 so is skipped.

□

4 Numerical examples

We specialize to a regime switching model. A discrete time finite state homogeneous Markov chain (MC) $J := (J_{t_n})_{n=0,1,\ldots,\infty}$ is defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_{t_n}\}, \mathbb{P})$ and it takes values in the state space $\mathcal{S} = \{\mathbf{0}, \mathbf{1}\}$ (which represents two states of the market, bull and bear). The $n-$step transition matrix $P^{(n)} = (P^n_{ij})$, is defined by

$$P^{(n)}_{ij} := \mathbb{P}(J_{t_n} = j | J_{t_0} = i), \quad i, j = \mathbf{0}, \mathbf{1} \qquad n = 0, 1, ..., \infty,$$

where $P^{(0)}_{ij} = 1$ when $i = j$, otherwise $P^{(0)}_{ij} = 0$. We assume that the distribution of J_0 is known, and

$$\mathbb{P}(J_0 = i | \mathcal{F}_0) = \mathbb{P}(J_0 = i), \qquad i = \mathbf{0}, \mathbf{1}.$$

The risk aversion is defined by $\gamma_{t_n} = \gamma(J_{t_n}.)$ In this section we give a concrete example; take the electricity industry one of the most weather-sensitive businesses in the economy. When the temperature increases there is a higher demand for electricity due to the usage of air conditioners. In turn this will lead to higher energy prices. In our model an energy provider hedges the weather exposure by selling one share of weather derivative to the representative agent. This is designed such that it has a higher payoff when temperature is high. The representative agent has an incentive to buy this product because of his/her income exposure to weather.

Consider an European call option on the temperature with a strike price $K = 10$; assume that $h = 0.3$, $C_0 = c = 10$, $S_0 = s = 10$ (this is normalized and it corresponds to 85 Fahrenheit degrees), $\rho = 0.5$, $\mu^c = 0.1$, $\sigma^c = 0.2$, $\mu^s = 0.3$, $\sigma^s = 0.50$. The energy price process (C_t) follows (2.2); the market price of risk (MPR) of the commodity (r^c_t) is defined

$$(r^c_{t_n})^2 = (\arctan(S_t) + \frac{\pi}{2}).$$

Therefore higher temperatures lead to an increased (MPR).

4.1 Single period

In this section, we present a numerical example of single-period, $N = 1$. Recall that in this case time consistent equilibrium coincides with time inconsistent equilibrium. Let $D_1 = (S_1 - K)^+$, and for simplicity assume $\varphi = 0$.

4.1.1 Equilibrium Price versus Indifference Price

The paths of the typical trajectories of the forward processes (C, S), are plotted in Fig.1, and the sample path of MPR in Fig.2.

It is easy, within our model, to compare numerically equilibrium price and indifference price of D_1.This is done in Fig. 3, Fig. 4 and Fig. 5, where we take $\rho = 0.5$ and $\gamma = 0.7$.

Next we introduce the income $I_1 = 7e^{-0.5(S_1 - s)h}$. Fig. 6 shows that the equilibrium price is an increasing function of the risk aversion γ. This is explained by an increase in

the hedging demand when the agent becomes more risk averse. This increase is due to weather impact on the income.

Next we add a nonspanned component to the income

$$I_1 = 7e^{-0.5(S_1-s)h} + 5e^{0.1h}1_{\{\Delta b_0^1=1,\Delta b_0^3=1\}} + 4e^{0.1h}1_{\{\Delta b_0^1=1,\Delta b_0^3=-1\}}+$$
$$+2e^{0.1h}1_{\{\Delta b_0^1=-1,\Delta b_0^3=1\}} + e^{0.1h}1_{\{\Delta b_0^1=-1,\Delta b_0^3=-1\}}.$$

Fig 7. shows the effect of this addition. The equilibrium price of the derivative is slightly lower with nonspanned income. This is explained by the fact that in this case only a part of the income is affected by weather, whence a lower hedging demand.

4.2 Two periods

Take $N = 2$, $D_2 = (S_2 - K)^+$, and $\varphi = 0$. Moreover

$$I_2 = 7e^{-0.5(S_2-s)h} + 5e^{0.1h}1_{\{\Delta b_1^1=1,\Delta b_1^3=1\}} + 4e^{0.1h}1_{\{\Delta b_1^1=1,\Delta b_1^3=-1\}}+$$
$$+2e^{0.1h}1_{\{\Delta b_1^1=-1,\Delta b_1^3=1\}} + e^{0.1h}1_{\{\Delta b_1^1=-1,\Delta b_1^3=-1\}}.$$

Fig. 8 plots the time inconsistent equilibrium price with and without unspanned income. The effect of the unspanned income becomes more pronounced with $N = 2$.
Fig. 9 plots the effect of time changing risk aversion. We are interested in the percentage change of the time consistent equilibrium price when the benchmark is the time inconsistent equilibrium price. We allowed the income to depend on the state of the economy.

$$I_2 = 7e^{(-0.5(S_1-s))h} + 10e^{0.03h}1_{\{J_0=\mathbf{0},\Delta b_1^3=1\}} + 8e^{0.03h}1_{\{J_0=\mathbf{0},\Delta b_1^3=-1\}}+$$
$$+5e^{0.03h}1_{\{J_0=\mathbf{1},\Delta b_1^3=1\}} + 4e^{0.03h}1_{\{J_0=\mathbf{1},\Delta b_1^3=-1\}}.$$

5 Appendix

5.1 Appendix A: Proof of Theorem 3.1

From (5.1) and (2.5), it follows that

$$\begin{aligned}
\mathbb{E}_{t_{N-1}}[-\exp(-\gamma_{t_{N-1}}(X^\pi_{t_N} + I_{t_N}))] &= \mathbb{E}_{t_{N-1}}[-\exp(-\gamma_{t_{N-1}}(X_{t_{N-1}} + \Delta X^\pi_{t_{N-1}} + I_{t_N}))] \\
&= -e^{-\gamma_{t_{N-1}}x}\mathbb{E}_{t_{N-1}}[\exp(-\gamma_{t_{N-1}}(\Delta X^\pi_{t_{N-1}} + I_{t_N}))] \\
&:= -e^{-\gamma_{t_{N-1}}x}g_{N-1}(\alpha,\beta,\cdot),
\end{aligned}$$

where

$$\Delta X^\pi_{t_{N-1}} := \alpha_{t_{N-1}}(\mu^c_{t_{N-1}}h + \sigma^c_{t_{N-1}}\sqrt{h}\Delta b^1_{t_{N-1}}) + \beta_{t_{N-1}}(\Delta D_{t_{N-1}} + \varphi(t_{N-1}, C_{t_{N-1}}, S_{t_{N-1}})h), \tag{5.1}$$

and

$$
\begin{aligned}
&g_{N-1}(\alpha,\beta,\cdot) := \mathbb{E}_{t_{N-1}}[\exp(-\gamma_{t_{N-1}}(\Delta X^{\pi}_{t_{N-1}} + I_{t_N}))] \\
&= \frac{1}{2} e^{-\gamma_{t_{N-1}}\alpha(\mu^c_{t_{N-1}}h+\sigma^c_{t_{N-1}}\sqrt{h})} \mathbb{E}_{t_{N-1}}\left[e^{-\gamma_{t_{N-1}}\beta(\Delta D_{t_{N-1}}+\varphi(t_{N-1},C_{t_{N-1}},S_{t_{N-1}})h)} e^{-\gamma_{t_{N-1}} I_{t_N}} | A_{t_{N-1}} \right] \\
&+ \frac{1}{2} e^{-\gamma_{t_{N-1}}\alpha(\mu^c_{t_{N-1}}h-\sigma^c_{t_{N-1}}\sqrt{h})} \mathbb{E}_{t_{N-1}}\left[e^{-\gamma_{t_{N-1}}\beta(\Delta D_{t_{N-1}}+\varphi(t_{N-1},C_{t_{N-1}},S_{t_{N-1}})h)} e^{-\gamma_{t_{N-1}} I_{t_N}} | A^c_{t_{N-1}} \right].
\end{aligned}
$$

Recall that $A_{t_k} := \{\Delta b^1_{t_k} = 1\}$ and $A^c_{t_k} := \{\Delta b^1_{t_k} = -1\}$. The function $g_{N-1}(\alpha,\beta,\cdot)$ has the following properties:

$$
g_{N-1}(0,0,\cdot) = \mathbb{E}_{t_{N-1}}[e^{-\gamma_{t_{N-1}} I_{t_N}}] \le 1;
$$

For a fixed β, it follows that for small h

$$
g_{N-1}(\infty,\beta,\cdot) = \infty;\ g_{N-1}(-\infty,\beta,\cdot) = \infty;
$$

By arbitrage argument it follows that $D_{t_{N-1}}$ belongs to the interval

$$
D_{t_{N-1}}(\omega) \in [\inf_Q \mathbb{E}^Q[D_{t_N}], \sup_Q \mathbb{E}^Q[D_{t_N}]],
$$

where Q ranges over the set of probability measures. Consequently,

$$
\Delta D_{t_{N-1}}(\omega) \in [D_{t_N} - \sup_Q \mathbb{E}^Q[D_{t_N}], D_{t_N} - \inf_Q \mathbb{E}^Q[D_{t_N}]].
$$

Thus, the sets $\{\omega : \Delta D_{t_{N-1}}(\omega) > 0\}$, and $\{\omega : \Delta D_{t_{N-1}}(\omega) < 0\}$ have positive probability. This implies that

$$
g_{N-1}(\alpha,\infty,\cdot) = \infty;\ g_{N-1}(\alpha,-\infty,\cdot) = \infty.
$$

From the above analysis, it follows that the minimum of the function of g is a critical point. First order conditions lead to

$$
\begin{aligned}
\frac{\partial g_{N-1}}{\partial \alpha} &= \frac{-\gamma_{t_{N-1}}(\mu^c_{t_{N-1}}h + \sigma^c_{t_{N-1}}\sqrt{h})}{2} \cdot e^{-\gamma_{t_{N-1}}\alpha(\mu^c_{t_{N-1}}h+\sigma^c_{t_{N-1}}\sqrt{h})} \\
&\quad \times \mathbb{E}_{t_{N-1}}\left[e^{-\gamma_{t_{N-1}}\beta(\Delta D_{t_{N-1}}+\varphi(t_{N-1},C_{t_{N-1}},S_{t_{N-1}})h)} e^{-\gamma_{t_{N-1}} I_{t_N}} | A_{t_{N-1}} \right] \\
&+ \frac{-\gamma_{t_{N-1}}(\mu^c_{t_{N-1}}h - \sigma^c_{t_{N-1}}\sqrt{h})}{2} \cdot e^{-\gamma_{t_{N-1}}\alpha(\mu^c_{t_{N-1}}h-\sigma^c_{t_{N-1}}\sqrt{h})} \\
&\quad \times \mathbb{E}_{t_{N-1}}\left[e^{-\gamma_{t_{N-1}}\beta(\Delta D_{t_{N-1}}+\varphi(t_{N-1},C_{t_{N-1}},S_{t_{N-1}})h)} e^{-\gamma_{t_{N-1}} I_{t_N}} | A^c_{t_{N-1}} \right] \\
&= 0,
\end{aligned} \tag{5.2}
$$

and

$$\frac{\partial g_{N-1}}{\partial \beta} = \frac{1}{2} e^{-\gamma_{t_{N-1}}\alpha(\mu^c_{t_{N-1}}h+\sigma^c_{t_{N-1}}\sqrt{h})} e^{-\gamma_{t_{N-1}}\beta\varphi(t_{N-1},C_{t_{N-1}},S_{t_{N-1}})h}$$

$$\times \mathbb{E}_{t_{N-1}}\Big[-\gamma_{t_{N-1}}(\Delta D_{t_{N-1}} + \varphi(t_{N-1}, C_{t_{N-1}}, S_{t_{N-1}})h) \cdot e^{-\gamma_{t_{N-1}}\beta\Delta D_{t_{N-1}}} e^{-\gamma_{t_{N-1}} I_{t_N}} | A_{t_{N-1}}\Big]$$

$$+\frac{1}{2} e^{-\gamma_{t_{N-1}}\alpha(\mu^c_{t_{N-1}}h-\sigma^c_{t_{N-1}}\sqrt{h})} e^{-\gamma_{t_{N-1}}\beta\varphi(t_{N-1},C_{t_{N-1}},S_{t_{N-1}})h}$$

$$\times \mathbb{E}_{t_{N-1}}\Big[-\gamma_{t_{N-1}}(\Delta D_{t_{N-1}} + \varphi(t_{N-1}, C_{t_{N-1}}, S_{t_{N-1}})h) \cdot e^{-\gamma_{t_{N-1}}\beta\Delta D_{t_{N-1}}} e^{-\gamma_{t_{N-1}} I_{t_N}} | A^c_{t_{N-1}}\Big]$$

$$= \quad 0. \tag{5.3}$$

Recall that

$$\Delta D_{t_{N-1}} + \varphi(t_{N-1}, C_{t_{N-1}}, S_{t_{N-1}})h = D_{t_N} - \mathbb{E}^{Q^*}_{t_{N-1}}[D_{t_N}], \tag{5.4}$$

for an equilibrium pricing measure Q^* to be found. Since $\mathbb{E}^{Q^*}_{t_{N-1}}[D_{t_N}]$ is $\mathcal{F}_{t_{N-1}}-$measurable, it follows that

$$\begin{aligned} \alpha^*_{t_{N-1}} &= \frac{1}{2\gamma(i)\sigma^c_{t_{N-1}}\sqrt{h}} \log\left(\frac{1+r^c_{t_{N-1}}\sqrt{h}}{1-r^c_{t_{N-1}}\sqrt{h}}\right) \\ &+ \frac{1}{2\gamma_{t_{N-1}}\sigma^c_{t_{N-1}}\sqrt{h}} \log\left(\frac{\mathbb{E}_{t_{N-1}}[e^{-\gamma_{t_{N-1}}\beta^*_{t_{N-1}}D_{t_N}} e^{-\gamma_{t_{N-1}}I_{t_N}} | A_{t_{N-1}}]}{\mathbb{E}_{t_{N-1}}[e^{-\gamma_{t_{N-1}}\beta^*_{t_{N-1}}D_N} e^{-\gamma(i)I_{t_N}} | A^c_{t_{N-1}}]}\right). \end{aligned}$$

By the equilibrium condition $\beta^*_{t_{N-1}} = 1$. This together with $\frac{\partial g_{N-1}}{\partial \beta} = 0$ lead to

$$\mathbb{E}_{t_{N-1}}\Big[(\Delta D_{t_{N-1}} + \varphi(t_{N-1}, C_{t_{N-1}}, S_{t_{N-1}})h) e^{-\gamma_{t_{N-1}}\Delta D_{t_{N-1}}} e^{-\gamma_{t_{N-1}}I_{t_N}} | A_{t_{N-1}}\Big] =$$

$$-e^{2\gamma_{t_{N-1}}\alpha^*_{t_{N-1}}\sigma^c_{t_{N-1}}\sqrt{h}} \mathbb{E}_{t_{N-1}}\Big[(\Delta D_{t_{N-1}}+\varphi(t_{N-1}, C_{t_{N-1}}, S_{t_{N-1}})h) e^{-\gamma_{t_{N-1}}\Delta D_{t_{N-1}}} e^{-\gamma_{t_{N-1}}I_{t_N}} | A^c_{t_{N-1}}\Big],$$

and

$$e^{2\gamma_{t_{N-1}}\alpha^*_{t_{N-1}}\sigma^c_{t_{N-1}}\sqrt{h}} = \frac{(1+r^c_{t_{N-1}}\sqrt{h})\mathbb{E}_{t_{N-1}}[e^{-\gamma_{t_{N-1}}D_{t_N}} e^{-\gamma_{t_{N-1}}I_{t_N}} | A_{t_{N-1}}]}{(1-r^c_{t_{N-1}}\sqrt{h})\mathbb{E}_{t_{N-1}}[e^{-\gamma_{t_{N-1}}D_{t_N}} e^{-\gamma_{t_{N-1}}I_{t_N}} | A^c_{t_{N-1}}]}. \tag{5.5}$$

Combing the above equations leads to

$$\begin{aligned} &\frac{2}{1-r^c_{t_{N-1}}\sqrt{h}} \mathbb{E}^{Q^*}_{t_{N-1}}[D_{t_N}] \\ =& \frac{\mathbb{E}_{t_{N-1}}[D_{t_N} e^{-\gamma_{t_{N-1}}D_{t_N}} e^{-\gamma_{t_{N-1}}I_{t_N}} | A_{t_{N-1}}]}{\mathbb{E}_{t_{N-1}}[e^{-\gamma_{t_{N-1}}D_{t_N}} e^{-\gamma_{t_{N-1}}I_{t_N}} | A_{t_{N-1}}]} \\ +& \frac{(1+r^c_{t_{N-1}}\sqrt{h})}{(1-r^c_{t_{N-1}}\sqrt{h})} \frac{\mathbb{E}_{t_{N-1}}[D_{t_N} e^{-\gamma_{t_{N-1}}D_{t_N}} e^{-\gamma_{t_{N-1}}I_{t_N}} | A^c_{t_{N-1}}]}{\mathbb{E}_{t_{N-1}}[e^{-\gamma(i)D_{t_N}} e^{-\gamma_{t_{N-1}}I_{t_N}} | A^c_{t_{N-1}}]}. \end{aligned}$$

This together with (5.4) imply that:

$$\begin{aligned} D_{t_{N-1}} - \varphi(t_{N-1}, C_{t_{N-1}}, S_{t_{N-1}})h &= \frac{1 - r^c_{t_{N-1}}\sqrt{h}}{2} \frac{\mathbb{E}_{t_{N-1}}[D_{t_N} e^{-\gamma_{t_{N-1}} D_{t_N}} e^{\gamma_{t_{N-1}} I_{t_N}} | A_{t_{N-1}}]}{\mathbb{E}_{t_{N-1}}[e^{-\gamma(i) D_{t_N}} e^{\gamma(i) I_{t_N}} | A_{t_{N-1}}]} \\ &+ \frac{1 + r^c_{t_{N-1}}\sqrt{h}}{2} \frac{\mathbb{E}_{t_{N-1}}[D_{t_N} e^{-\gamma_{t_{N-1}} D_{t_N}} e^{\gamma_{t_{N-1}} I_{t_N}} | A^c_{t_{N-1}}]}{\mathbb{E}_{t_{N-1}}[e^{-\gamma_{t_{N-1}} D_{t_N}} e^{\gamma_{t_{N-1}} I_{t_N}} | A^c_{t_{N-1}}]}. \end{aligned}$$

Thus, the equilibrium price is

$$D_{t_{N-1}} = \mathbb{E}_{t_{N-1}}[(D_{t_N} + \varphi(t_{N-1}, C_{t_{N-1}}, S_{t_{N-1}})h)\Lambda_{t_N}],$$

where Λ_{t_N} was defined in (3.2).

□

5.2 Appendix B: Proof of Theorem 3.2

We will prove the result for time consistent equilibrium; the proof for time inconsistent equilibrium is similar and hence omitted. First consider the time period is $[(N-n)h, (N-n+1)h)$ and choose an arbitrary control $\pi = (\alpha, \beta)$ for any $J_{t_n} \in \{\mathbf{0}, \mathbf{1}\}$ as follows:

$$\pi = \begin{cases} \pi^*_{t_n}, & \text{for } n = N-(n-1), N-(n-2)\cdots N-1, \\ \pi_{t_n}, & \text{for } n = N-n. \end{cases} \tag{5.6}$$

For convenience, denote $\varphi_{t_{N-n}} = \varphi(t_{N-n}, C_{t_{N-n}}, S_{t_{N-n}})$. Assume $J_{t_{N-n}} = i$. From

$$X^\pi_{t_N} = X^\pi_{t_{N-(n-1)}} + \sum_{k=N-(n-1)}^{N-1} \Delta X^*_{t_k},$$

it follows that

$$\begin{aligned} &\mathbb{E}_{t_{N-n}}[-\exp(-\gamma_{t_{N-n}}(X^\pi_{t_N} + I_{t_N})] \\ = \quad & -e^{-\gamma_{t_{N-n}} x} \mathbb{E}_{t_{N-n}}\Big[- e^{-\gamma_{t_{N-n}} \Delta X^\pi_{t_{N-n}}} \cdot \mathbb{E}_{t_{N-n+1}}[e^{-\gamma_{t_{N-n}}[\sum_{k=N-(n-1)}^{N-1} \Delta X^*_{t_k} + I_{t_N}]}]\Big] \\ = \quad & \mathbb{E}_{t_{N-n}}[-e^{-\gamma_{t_{N-n}} \Delta X^\pi_{t_{N-n}}} \cdot e^{-\gamma_{t_{N-n}} Y^*_{t_{N-(n-1)}}}] \\ := \quad & -e^{-\gamma_{t_{N-n}} x} g_{N-n}(\alpha, \beta, \cdot). \end{aligned}$$

Here for $k = N-(n-1), \cdots, N-2, N-1$,

$$\Delta X^*_{t_k} = \alpha^*_{t_k}(\mu^c_{t_k} h + \sqrt{h}\sigma^c_{t_k}\Delta b^1_{t_k}) + \Delta D_{t_k} + \varphi(t_k, C_{t_k}, S_{t_k})h,$$

and

$$\begin{aligned}
&g_{N-n}(\alpha,\beta,\cdot) = \mathbb{E}_{t_{N-n}}[-e^{-\gamma_{t_{N-n}}\Delta X^\pi_{t_{N-n}}} \cdot e^{-\gamma_{t_{N-n}}Y^*_{t_{N-(n-1)}}}] \\
&= \frac{1}{2}e^{-\gamma_{t_{N-n}}\alpha(\mu^c_{t_{N-n}}h+\sigma^c_{t_{N-n}}\sqrt{h})}\mathbb{E}_{t_{N-n}}\left[e^{-\gamma_{t_{N-n}}\beta(\Delta D_{t_{N-n}}+\varphi_{t_{N-n}}h)}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A_{t_{N-n}}\right] \\
&+\frac{1}{2}e^{-\gamma_{t_{N-n}}\alpha(\mu^c_{t_{N-n}}h-\sigma^c_{t_{N-n}}\sqrt{h})}\mathbb{E}_{t_{N-n}}\left[e^{-\gamma_{t_{N-n}}\beta(\Delta D_{t_{N-n}}+\varphi_{t_{N-n}}h)}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A^c_{t_{N-n}}\right]
\end{aligned}$$

with $A_{t_{N-n}} := \{\Delta b^1_{t_{N-n}} = 1\}$ and $A^c_{t_{N-n}} := \{\Delta b^1_{t_{N-n}} = -1\}$. Arguing as in the one period case we get

$$g_{N-n}(0,0,\cdot) = \mathbb{E}_{t_{N-n}}[e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}] \leq \infty;$$

$$g_{N-n}(\infty,\beta,\cdot) = \infty;\ g_{N-n}(-\infty,\beta,\cdot) = \infty;$$

From arbitrage considerations it follows that

$$D_{t_{N-n}}(\omega) \in [\inf_Q \mathbb{E}^Q[D_{t_{N-n+1}}], \sup_Q \mathbb{E}^Q[D_{t_{N-n+1}}]],$$

where Q is the set of probability measures. Thus

$$\Delta D_{t_{N-n}}(\omega) \in [D_{t_{N-n+1}} - \sup_Q \mathbb{E}^Q[D_{t_{N-n+1}}], D_{t_{N-n+1}} - \inf_Q \mathbb{E}^Q[D_{t_{N-n+1}}]],$$

so the sets: $\{\omega : \Delta D_{t_{N-n}}(\omega) > 0\}$, and $\{\omega : \Delta D_{t_{N-n}}(\omega) < 0\}$ have positive probability. Consequently, it follows that:

$$g_{N-n}(\alpha,\infty,\cdot) = \infty;\ g_{N-n}(\alpha,-\infty,\cdot) = \infty.$$

Therefore the minimum of $g_{N-n}(\alpha,\beta,\cdot)$ is a critical point. Hence

$$\begin{aligned}
\frac{\partial g_{N-n}}{\partial\alpha} &= \frac{(\mu^c_{t_{N-n}}h + \sigma^c_{t_{N-n}}\sqrt{h})}{2}e^{-\gamma_{t_{N-n}}\alpha(\mu^c_{t_{N-n}}h+\sigma^c_{t_{N-n}}\sqrt{h})} \\
&\quad\times\mathbb{E}_{t_{N-n}}\left[e^{-\gamma_{t_{N-n}}\beta^*(\Delta D_{t_{N-n}}+\varphi_{t_{N-n}}h)}e^{-\gamma(i)Y^*_{t_{N-n+1}}}|A_{t_{N-n}}\right] \\
&+ \frac{(\mu^c_{t_{N-n}}h - \sigma^c_{t_{N-n}}\sqrt{h})}{2}e^{-\gamma_{t_{N-n}}\alpha(\mu^c_{t_{N-n}}h-\sigma^c_{t_{N-n}}\sqrt{h})} \\
&\quad\times\mathbb{E}_{t_{N-n}}\left[e^{-\gamma_{t_{N-n}}\beta^*(\Delta D_{t_{N-n}}+\varphi_{t_{N-n}}h)}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A^c_{t_{N-n}}\right] \\
&= 0. \qquad (5.7)
\end{aligned}$$

By direct calculation, we get that the optimal time consistent trading strategy is

$$
\begin{aligned}
& \alpha^*_{t_{N-n}} = \frac{1}{2\gamma_{t_{N-n}}\sigma^c_{t_{N-n}}\sqrt{h}} \log[\frac{1+r^c_{t_{N-n}}\sqrt{h}}{1-r^c_{t_{N-n}}\sqrt{h}}] \\
+ \quad & \frac{1}{2\gamma_{t_{N-n}}\sigma^c_{t_{N-n}}\sqrt{h}} \log(\frac{\mathbb{E}_{t_{N-n}}[e^{-\gamma_{t_{N-n}}\beta^*_{t_{N-n}}D_{t_{N-n+1}}}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A_{t_{N-n}}]}{\mathbb{E}_{t_{N-n}}[e^{-\gamma_{t_{N-n}}\beta^*_{t_{N-n}}D_{t_{N-n+1}}}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A^c_{t_{N-n}}]}).
\end{aligned}
\tag{5.8}
$$

From the equilibrium conditions it follows that $\beta^*_{t_{N-n}} = 1$. This combined with $\frac{\partial g_{N-n}}{\partial \beta} = 0$, yield the equilibrium price at T_{N-n}. First, from $\frac{\partial g_{N-n}}{\partial \beta} = 0$, one gets

$$
\begin{aligned}
& \mathbb{E}_{t_{N-n}}\left[(\Delta D_{t_{N-n}} + \varphi_{t_{N-n}}h)\cdot e^{-\gamma_{t_{N-n}}(\Delta D_{t_{N-n}}+\varphi_{t_{N-n}}h)}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A_{t_{N-n}}\right] \\
= & -e^{2\gamma_{t_{N-n}}\alpha\sigma^c_{t_{N-n}}\sqrt{h}}\mathbb{E}_{t_{N-n}}\left[(\Delta D_{t_{N-n}} + \varphi_{t_{N-n}}h)\cdot e^{-\gamma_{t_{N-n}}(\Delta D_{t_{N-n}}+\varphi_{t_{N-n}}h)}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A^c_{t_{N-n}}\right].
\end{aligned}
$$

From $\frac{\partial g_{N-n}}{\partial \alpha} = 0$ it follows that

$$
e^{2\gamma_{t_{N-n}}\alpha^*\sigma^c_{t_{N-n}}\sqrt{h}} = \frac{(1+r^c_{t_{N-n}}\sqrt{h})\mathbb{E}_{t_{N-n}}[e^{-\gamma_{t_{N-n}}\beta^* D_{t_{N-n+1}}}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A_{t_{N-n}}]}{(1-r^c_{t_{N-n}}\sqrt{h})\mathbb{E}_{t_{N-n}}[e^{-\gamma(i)\beta^* D_{t_{N-n+1}}}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A^c_{t_{N-n}}]}
$$

This together with

$$
\Delta D_{t_{N-n}} + \varphi_{t_{N-n}}h = D_{t_{N-n+1}} - \mathbb{E}^{Q^*}_{t_{N-n}}[D_{t_{N-n+1}}],
$$

(here Q^* is the equilibrium probability measure to be found) yield

$$
\begin{aligned}
& (\frac{2}{1-r^c_{t_{N-n}}\sqrt{h}})\mathbb{E}^{Q^*}_{t_{N-n}}[D_{t_{N-n+1}}] \\
= \quad & \frac{\mathbb{E}_{t_{N-n}}[D_{t_{N-n+1}}e^{-\gamma_{t_{N-n}}D_{t_{N-n+1}}}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A_{t_{N-n}}]}{\mathbb{E}_{t_{N-n}}[e^{-\gamma_{t_{N-n}}D_{t_{N-n+1}}}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A_{t_{N-n}}]} \\
& + \frac{(1+r^c_{t_{N-n}}\sqrt{h})}{(1-r^c_{t_{N-n}}\sqrt{h})}\frac{\mathbb{E}_{t_{N-n}}[D_{t_{N-n+1}}e^{-\gamma_{t_{N-n}}D_{t_{N-n+1}}}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A^c_{t_{N-n}}]}{\mathbb{E}_{t_{N-n}}[e^{-\gamma_{t_{N-n}}D_{t_{N-n+1}}}e^{-\gamma_{t_{N-n}}Y^*_{t_{N-n+1}}}|A^c_{t_{N-n}}]}.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
D_{t_{N-n}} - \varphi_{t_{N-n}} h &= \frac{1 - r^c_{t_n}\sqrt{h}}{2} \frac{\mathbb{E}_{t_{N-n}}[D_{t_{N-n+1}} e^{-\gamma_{t_{N-n}} D_{t_{N-n+1}}} e^{-\gamma_{t_{N-n}} Y^*_{t_{N-n+1}}} | A_{t_{N-n}}]}{\mathbb{E}_{t_{N-n}}[e^{-\gamma_{t_{N-n}} D_{t_{N-n+1}}} e^{-\gamma_{t_{N-n}} Y^*_{t_{N-n+1}}} | A_{t_{N-n}}]} \\
+ & \quad \frac{1 + r^c_{t_n}\sqrt{h}}{2} \frac{\mathbb{E}_{t_{N-n}}[D_{t_{N-n+1}} e^{-\gamma_{t_{N-n}} D_{t_{N-n+1}}} e^{-\gamma_{t_{N-n}} Y^*_{t_{N-n+1}}} | A^c_{t_{N-n}}]}{\mathbb{E}_{t_{N-n}}[e^{-\gamma_{t_{N-n}} D_{t_{N-n+1}}} e^{-\gamma_{t_{N-n}} Y^*_{t_{N-n+1}}} | A^c_{t_{N-n}}]}.
\end{aligned} \tag{5.9}
$$

Thus, the equilibrium price is

$$
D_{t_{N-n}} \quad = \quad \mathbb{E}_{t_{N-n}}[(D_{t_{N-n+1}} + \varphi_{t_{N-n}} h)\Lambda^*_{t_{N-n+1}}], \tag{5.10}
$$

with $\Lambda^*_{t_{N-n+1}}$ defined in (3.11).

5.3 Appendix C: Proof of Corollary 3.2

We consider the time period $[(N-n)h, (N-n+1)h)$. Recall that

$$
\hat{X}_{t_N} = x + \Delta \hat{X}_{t_{N-n}} + \sum_{k=N-(n-1)}^{N-1} \Delta \hat{X}_{t_k},
$$

where

$$
\Delta \hat{X}_{t_k} = \hat{\alpha}_{t_k}(\mu^c_{t_k} h + \sqrt{h}\sigma^c_{t_k} \Delta b^1_{t_k}) + \hat{\beta}_{t_k}(\Delta D_{t_k} + \varphi(t_k, C_{t_k}, S_{t_k})h),
$$

for $k = N-(n-1), \cdots, N-2, N-1$. Thus, we have

$$
\begin{aligned}
& \mathbb{E}_{t_{N-n}}(U'(\hat{W}_{t_N})) \\
&= \mathbb{E}_{t_{N-n}}[-e^{-\gamma \Delta \hat{X}_{t_{N-n}}} \cdot e^{-\gamma \hat{Y}_{t_{N-(n-1)}}}] \\
&= \frac{1}{2} e^{-\gamma \hat{\alpha}_{t_{N-n}}(\mu^c_{t_{N-n}} h + \sigma^c_{t_{N-n}} \sqrt{h})} e^{-\gamma \varphi_{t_{N-n}} h} \mathbb{E}_{t_{N-n}}\left[e^{-\gamma \Delta D_{t_{N-n}}} e^{-\gamma \hat{Y}_{t_{N-n+1}}} | A_{t_{N-n}}\right] \\
&+ \frac{1}{2} e^{-\gamma \hat{\alpha}_{t_{N-n}}(\mu^c_{t_{N-n}} h - \sigma^c_{t_{N-n}} \sqrt{h})} e^{-\gamma \varphi_{t_{N-n}} h} \mathbb{E}_{t_{N-n}}\left[e^{-\gamma \Delta D_{t_{N-n}}} e^{-\gamma \hat{Y}_{t_{N-n+1}}} | A^c_{t_{N-n}}\right].
\end{aligned}
$$

From direct calculations, it follows that on the set $\{\omega : \omega \in A^c_{t_{N-n}}\}$

$$\begin{aligned}
&\frac{\mathbb{E}_{t_{N-n+1}}[U'(\hat{W}_{t_N})]}{\mathbb{E}_{t_{N-n}}[U'(\hat{W}_{t_N})]} \\
= &\frac{(1+r^c_{t_{N-n}}\sqrt{h})e^{\gamma\hat{\alpha}_{t_{N-n}}\sigma^c_{t_{N-n}}\sqrt{h}}e^{-\gamma\hat{\alpha}_{t_{N-n}}\sigma^c\sqrt{h}\Delta b^1_{t_{N-n}}}e^{-\gamma D_{t_{N-n+1}}}e^{-\gamma\hat{Y}_{t_{N-n+1}}}}{\mathbb{E}_{t_{N-n}}\left[e^{-\gamma D_{t_{N-n+1}}}e^{-\gamma\hat{Y}_{t_{N-n+1}}}|A^c_{t_{N-n}}\right]} \\
= &\lambda_{t_{N-n}}\frac{e^{-\gamma D_{t_{N-n+1}}}e^{-\gamma\hat{Y}_{t_{N-n+1}}}}{\mathbb{E}_{t_{N-n}}[e^{-\gamma D_{t_{N-n+1}}}e^{-\gamma\hat{Y}_{t_{N-n+1}}}|A^c_{t_{N-n}}]}.
\end{aligned}$$

Moreover, on the set of $\{\omega : \omega \in A_{t_{N-n}}\}$

$$\frac{\mathbb{E}_{t_{N-n+1}}[U'(\hat{W}_{t_N})]}{\mathbb{E}_{t_{N-n}}[U'(\hat{W}_{t_N})]} = \lambda_{t_{N-n}}\frac{e^{-\gamma D_{t_{N-n+1}}}e^{-\gamma\hat{Y}_{t_{N-n+1}}}}{\mathbb{E}_{t_{N-n}}[e^{-\gamma D_{t_{N-n+1}}}e^{-\gamma\hat{Y}_{t_{N-n+1}}}|A_{t_{N-n}}]}.$$

Therefore

$$\frac{\mathbb{E}_{t_{N-n+1}}[U'(\hat{W}_{t_N})]}{\mathbb{E}_{N-n}[U'(\hat{W}_{t_N})]} = \hat{\Lambda}_{t_{N-n+1}}. \tag{5.11}$$

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